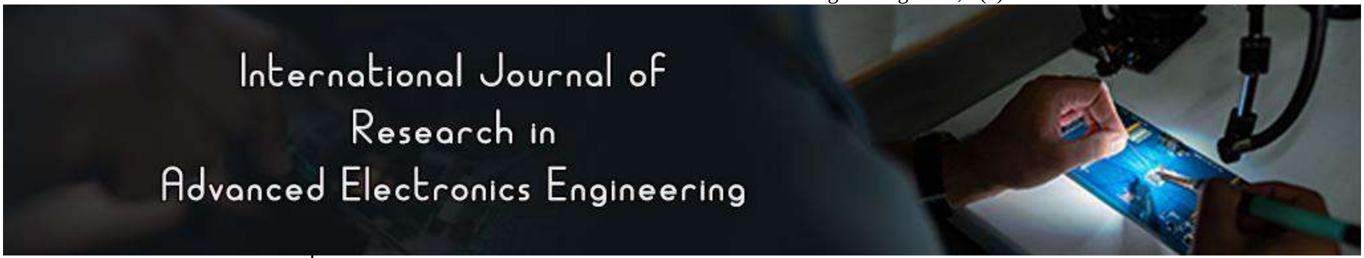

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Estimation of specific class, in the unit disc holomorphic functions

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Abstract
Let *f* be in the unit Disk holomorphic function with *Re (f (z)* > 0 and f (0) =1. For this type of function, we studied the boundary estimate values in the unit Disk and obtained the best estimation. The Schwarz lemma, the Schwarz-pick lemma, and the Green function for a unit Disk are all examples of lemmas. The equality of *f* and *g* in unit disc restricted holomorphic functions under specific conditions, as well as the relationships between the Green function for a unit disc and the Schwarz lemma, are useful tools in this paper to achieve the goal. Moreover we have obtained that *f* is, *subordinate* to 1+Z-1-Z.

Keywords: Schwarz lemma, holomorphic function, Green function, subordinate function, auto morphisms

Introduction

The values of certain classes of holomorphic functions that are definite from the unit disc to the unit disc are estimated by Schwarz lemma, and Schwarz-pick lemma states that the distance between certain functions f decreases in the pseudo hyperbolic metric. The estimate value for a specific class of holomorphic functions and their derivatives is obtained using the aforementioned properties and the same useful relation.

Lemma 1: (Schwarz lemma) Let D be the open unit disk in $\mathbb{C}$ and $f: D \to D$ be a holomorphic function with $f(0) = 0, |f(z)| \leq 1$ then

i). $|f(z)| \leq |z|$ for $z \in D$.

ii). $|f'(0)| \leq 1$.

(here $|f(z)| := \sup_{z \in D} |f(z)|$)

If for a $z_0 \in D \backslash \{0\}, |f(z_0)| = |z_0|$ or $|f'(0)| = 1$, then $f(z) = cz$ with $|c| = 1$.

Proof

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. It is clear that $a_1 = f'(0)$. Then $h(z) = \frac{f(z)}{z}$ is also holomorphic in D.

For $z_0 \in D\backslash \ \{0\}$ with $0 < |z_0| < r < 1$

$$|h(z_0)| \leq \max_{|z|=r} |h(z)| = \max_{|z|=r} \left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$$

For $r \to 1$ we have $|h(z_0)| \leq 1 \Rightarrow |f(z_0)| \leq |z_0|$.

$$|h(0)| = |a_1| \Rightarrow |f'(0)| \leq 1$$.

For the equality we have:
$|f(z_0)| = |z_0| \Longrightarrow |h(z_0)| = 1$, From the maximum principle $h(z) = a \in \partial D$.
$|f'(0)| = 1 \Longrightarrow |h(0)| = 1$, also from the maximum principle $h(z) = a \in \partial D$.

2. Research Method

The Schwarz lemma, Schwarz-pick lemma, Corollary (2), and a specific type of Mobius transformation, Green Function for a Domain are used as important tools in this research paper. The Schwarz and Schwarz-Pic lemmas, as well as their properties and relationship to the pseudo hyperbolic metric, are introduced first. Theorem 5 and corollary 6 show that, in the unit disk restricted function with infinite many equal zeros, is unique. Using the previously mentioned concepts and the Green function for unit disk, a boundary estimate value in the unit holomorphic function with positive real part and $f(0) = 1$ was obtained.

Schwarz lemma extension

Let f be holomorph in D, 0 is a n times zero of f and $|f(z)| \le 1$ for $z \in D$. Then
i. $|f(z)| \le |z|^n$ for $\in D$.
ii. $|f^{(n)}(0)| \le n!$.

If for a $z_0 \in D\backslash\{0\}$, $|f(z_0)| = |z_0|^n$ or $|f^{(n)}(0)| = n!$, then $f(z) = cz^n$ with $|c| = 1$,

To prove the above relations we use $f(z) = z^n g(z)$ with $g(z)$ is holomorph in D, and easily the result obtained.
Corollary 2 the $g: D \rightarrow D$ bijective and holomorphic functions have the following form

$$g(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z}, |a| < 1 .$$

This kind of functions call auto morphisms of unit desk. The set of these functions show with $Aut\ D$.
For $|z| = 1$ we have

$$|g(z)|^2 = \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{\bar{z}-\bar{a}}{1-a\bar{z}}\right) = \frac{|z|^2 - a\bar{z} - \bar{a}z + |a|^2}{1-\bar{a}z - a\bar{z} + |a|^2|z|^2} = 1$$

So that $|g(z)| = 1$on the boundary. For this reason the function g can be used, in variations of Schwarz Lemma, to solve extremal problems for analytic functions .
Proof
let $g \in Aut\ D$ with $g(0) = 0 \Longrightarrow g^{-1}(0) = 0$. From the Schwarz lemma $|g(z)| \le |z|$ and also $|g^{-1}(z)| \le |z|$ for $|z| < 1$. Therefor $|g(z)| = |z|$ for $|z| < 1$. For the second time apply the Schwarz lemma obtains the $g(z) = e^{i\varphi}z$.
Let $f(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z}$, $|a| < 1$. From a simple calculation $|f(z)| = |1|$ for $|z| = 1$.

Let $g \in Aut\ D$ with $g(a) = 0$. Then $h = g \circ f^{-1} \in Aut\ D$ with $h(0) = 0$ it means $g(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z}$.

Lemma 3 (Schwarz-Pick)

Let $f: D \rightarrow D$ be holomorphic, $z_0 \in D$. Then for all $z \in D$

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \le \left|\frac{z-z_0}{1-\overline{z_0}z}\right|.$$

Remark: the expression

$$\varphi(z, z_0) = \left|\frac{z-z_0}{1-\overline{z_0}z}\right|$$

Is the pseudo hyperbolic metric on the disk. Thus Schwarz-pick lemma says that the function f distance decreasing in the pseudo hyperbolic metric.

Proof

For $\alpha \in D$ we define a function $F: D \rightarrow D$ with

$$F(\alpha) := \frac{f\left(\frac{\alpha+z_0}{1+\bar{z}_0\alpha}\right) - f(z_0)}{1-\overline{f(z_0)}f\left(\frac{\alpha+z_0}{1+\bar{z}_0\alpha}\right)}$$

It is clear that $F(0) = 0$. From the Schwarz Lemma $|F(\alpha)| \le |\alpha|$ for all $\alpha \in D$. Also for all $z \in D$

$$\left|F\left(\frac{z-z_0}{1-\bar{z}_0z}\right)\right| \le \left|\frac{z-z_0}{1-\bar{z}_0z}\right|.$$

And there for

$$\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \le \left|\frac{z-z_0}{1-\bar{z}_0z}\right|.$$

Corollary 4: Let $f: D \rightarrow D$ be holomorph. Then for all $z \in D$
$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}$.

Proof
From the Schwarz-pick lemma we have

$$\left|\frac{f(v)-f(z)}{v-z}\right| \le \left|\frac{1-\overline{f(z)}f(v)}{1-\bar{z}v}\right|,$$

$$|f'(z)| = \left|\lim_{v\to z}\frac{f(v)-f(z)}{v-z}\right|$$

$\leq \left|\lim_{v\to z} \frac{1-\overline{f(z)}f(v)}{1-\bar{z}v}\right| \xrightarrow{v\to z} \frac{1-|f(z)|^2}{1-|z|^2}$.

Theorem 5: Let $f: D \to D$ be holomorphic and $z_k \in D, k = 1,2,3,\ldots$ with $f(z_k) = 0$, moreover f is not identity zero in D. Then

$\sum_{k=1}^{\infty} 1 - |z_k| < \infty$.

It means that the zeros points of f have to be near to ∂D .

Proof: assume that $f(0) \neq 0$ (otherwise we can study $f(z)/z^p$ for a suitable $p \in \mathbb{N}$)
For $n \in \mathbb{N}$

$g_n(z) \coloneqq \prod_{k=1}^{n} \frac{z-z_k}{1-\bar{z}_k z}$,

Then

$\left|\frac{f(z)}{g_1(z)}\right| = \left|\frac{f(z)}{\frac{z-z_1}{1-\bar{z}_1 z}}\right| = \left|\frac{\frac{f(z)-f(z_1)}{1-\overline{f(z_1)}f(z)}}{\frac{z-z_1}{1-\bar{z}_1 z}}\right| \leq 1$

$h_1 \coloneqq \frac{f(z)}{g_1(z)}: D \to D$, is holomorphic in D. From the Schwarz lemma, for all $z \in D$

$\left|\frac{h_1(z)}{\frac{z-z_2}{1-\bar{z}_2 z}}\right| \leq 1,$

Therefore $\frac{f(z)}{g_n(z)}: D \to D$ is holomorphic in D for every $n \in \mathbb{N}$. Specialize for $n \in \mathbb{N}$

$|f(0)| \leq |g_n(0)| = \prod_{k=1}^{n} |z_k|$

Since the $0 < x < 1 \Rightarrow 1 - x \leq -\log x$ we have

$\sum_{k=1}^{\infty} 1 - |z_k| \leq \sum_{k=1}^{n} -\log|z_k|$

$= \log \prod_{k=1}^{n} \frac{1}{|z_k|} \leq \log \frac{1}{|f(0)|}$.

Corollary 6
Let f *and* g be holomorphic, restricted in D and exist a sequence $\{z_k\}_{k\in\mathbb{N}} \subset D$ with

$\sum_{k=1}^{\infty} 1 - |z_k| = \infty$ and

$f(z_k) = g(z_k)$ for all $k \in \mathbb{N}$,

Then $f \equiv g$ in D .

Proof: Let $|f| \leq T_1$, $|g| \leq T_2$ in D. Then $T \coloneqq f - g$ is holomorphic in D with $T(z_k) = 0$ for all $k \in \mathbb{N}$ and $|T(z)| \leq T_1 + T_2 =: S$ it means that T is restricted in D. Moreover $\tilde{T}(z) \coloneqq \frac{T(z)}{S}: D \to D$ is holomorphic, have to be Zero. Then $f \equiv g$ in D.

Green Function for a disc
Let's Ω a disk, $g: \Omega \times \Omega \to \bar{\bar{\mathbb{R}}}$ is Green function for Ω, if
i). For all $z_0 \in \Omega$ the $g(z, z_0)$ is harmonic in $\Omega \backslash \{z_0\}$.
ii). For all $z_0 \in \Omega$ the $g(z, z_0) + \log|z - z_0|$ is harmonic in $\Omega \backslash \{z_0\}$.
iii). For all $z_0 \in \Omega$ the $\lim_{z\to\xi\partial D} g(z, z_0) = 0$.

Corollary 7. for every Ω disk exist at most one Green function.

Proof. Let's $z_0 \in \Omega$ and $C \subset \Omega$ an arbitrary circle with center z_0. g is in $\Omega\backslash C$ holomorphic, if C is enough small then $g(z, z_0) > 0$ for $z \in \partial C$. From the minimal principle for the harmonic functions whit $\lim_{z\to\xi\partial D} g(z, z_0) = 0$, implies that $g(z, z_0) > 0$ for all $z \in \Omega\backslash C$. Since $C \subset \Omega$ is arbitrary circle the assertion obtained.

Corollary 8: Let $\Omega \subset \mathbb{C}$ a is connected disk, $f: \Omega \to D$ bijective and holomorphic. Then

$g(z, z_0) \coloneqq -\log\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)}\right|$

Is the green function for Ω.

Proof.
a) f is injective then $g(z, z_0)$ is harmonic in $\Omega\backslash\{z_0\}$.
b $g(z, z_0) + \log|z - z_0| = -\log\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| + \log\left|1 - \overline{f(z_0)}f(z)\right|$ is harmonic in Ω (because $\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)} \neq 0$, since $f' \neq 0$).
$z \to \partial D \Longrightarrow f(z) \to e^{i\theta} \Longrightarrow g(z, z_0) \longrightarrow -\log \underbrace{\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right|}_{=1} = 0$.

Problem
let's f is holomorphic in D with $Re(f(z)) > 0$ and $f(0) = 1$ then

f is subordinate of $\frac{1+z}{1-z}$.
$\frac{1-|z|}{1+|z|} \leq |f(z)| \leq \frac{1+|z|}{1-|z|}$.
Proof (a) Since $\frac{1+z}{1-z}$ is a Mobius transformation, which unit cercal map holomorphic to upper half plane and 0 to 1 then (a) obtained.
(b) Let W is a subordinate function. Then

$$\frac{1-|z|}{1+|z|} \leq \frac{1-|w(z)|}{1+|w(z)|} \leq \frac{1+|w(z)|}{1-|w(z)|} = |f(z)| \leq \frac{1+|w(z)|}{1-|w(z)|} \leq \frac{1+|z|}{1-|z|}$$

IV Conclusion

i. Let's f subordinate of g. Then for all $0 < r \leq 1$ $f(D_r(0)) \subset g(D_r(0)$.

Because Let $0 < r \leq 1$ given and Let $|z| < r$. From the subordinate function w and Schwarz lemma

$$|w(z)| \leq |z| < r$$

Therefor $w(z) \in K_r(0)$ thus

$$f(z) = g(w(z)) \in g(K_r(0))$$

ii. Let $\Omega = \mathrm{D}$, $f(z) = z$. Then $-log\left|\frac{z-z_0}{1-\overline{z_0}z}\right|$ is the green function for D. From the Linelof principle for every $f: D \to D$ holomorphic function

$$-log\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \geq -log\left|\frac{z-z_0}{1-\overline{z_0}z}\right|$$

Then

$$log\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \leq log\left|\frac{z-z_0}{1-\overline{z_0}z}\right| \Longrightarrow \left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \leq \left|\frac{z-z_0}{1-\overline{z_0}z}\right|$$

iii. This state the fact that Schwarz had proved classic method.

From the above study we can obtained the classical Ahlfors Lemma.
Let $f: D \to D$ be a holomorphic function. Then for all $z_1, z_2 \in D$

$$\gamma(f(z_1), f(z_2) \leq \gamma(z_1, z_2)$$

(here γ is hyperbolic metric). It means, that the distance according to hyperbolic metric between the map of two points of a holomorphic functions inside the unit disk become smaller. Since:

From the Schwarz-Pick lemma we have

$$\frac{f'(z)}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \quad z \in D$$

iv. Let γ be non-Euclid's distance from z_1 to z_2 then

v.

$$\gamma(f(z_1), f(z_2)) \leq L_h(f \circ \lambda) = \int_0^1 \frac{2|f'(\lambda(t))\lambda'(t)|}{1-|f(\lambda(t))|^2} dt \leq \int_0^1 \frac{2|\lambda'(t)|}{1-|\lambda(t)|^2} dt = \gamma(z_1, z_2)$$

This is a formation of classical Ahlfors Lemma.

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